

ANALYSIS OF PERFECTLY MATCHED LAYER OPERATORS FOR ACOUSTIC SCATTERING ON MANIFOLDS WITH QUASICYLINDRICAL ENDS

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ABSTRACT. In this paper we prove stability and exponential convergence of the Perfectly Matched Layer (PML) method for acoustic scattering on manifolds with axial analytic quasicylindrical ends. These manifolds model long-range geometric perturbations (e.g. bending or stretching) of tubular waveguides filled with homogeneous or inhomogeneous media.

We construct non-reflective infinite PMLs replacing the metric on a part of the manifold by a non-degenerate complex symmetric tensor field. We prove that the problem with PMLs of finite length is uniquely solvable and solutions to this problem locally approximate scattered solutions with an error that exponentially tends to zero as the length of PMLs tends to infinity.

Keywords: Acoustic Scattering, Non-selfadjoint Operators, Essential Spectrum, Compound Expansions, PML, Inhomogeneous Media

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1. INTRODUCTION

The motivation of this work comes from the problem of numerical modeling of acoustic scattering in tubular waveguides geometrically perturbed up to infinity (e.g. bent or stretched) and filled with homogeneous or inhomogeneous media. In order to obtain a good approximation of scattered waves by numerical solutions of a problem with finite computational domain, waveguides should be truncated without creating excessive reflections from the artificial boundary of truncation. The idea is to place in front of the boundary of truncation a layer strongly absorbing the scattered waves. Due to the strong attenuation, the homogeneous Dirichlet boundary condition is a suitable boundary condition on the boundary of truncation. This truncation scheme supplemented with very special construction of the layer is widely known as the Perfectly Matched Layer (PML) method, originally introduced in [1]. The method is in common use for numerical analysis of a wide class of problems. For some of them, stability and convergence of the method have been proved mathematically; e.g. [2, 5, 7, 8, 11, 14]. In the present paper we develop the PML method for Neumann Laplacians on manifolds with axial analytic quasicylindrical ends and prove stability and exponential convergence of the method. Neumann Laplacians model the scattering problem described in the beginning of introduction, see, e.g., [3].

As is known, construction of PMLs is closely related to complex scaling. Complex scaling involves complex dilation of variables and has a long tradition in mathematical physics and numerical analysis [4, 16]. In this paper we construct PMLs in a different way. Instead of complex dilation of variables we replace the metric on a part of the manifold by a non-degenerate complex symmetric tensor field. This approach is close in spirit to [14] and can be understood as a deformation of the Riemannian geometry by means of the complex scaling. As a result of this deformation all formulas for the quadratic form and for coordinate representations of the Neumann Laplacian turn into the corresponding formulas for the quadratic form and the non-selfadjoint operator describing infinite PMLs. This essentially simplifies construction and tractability of the formulas. Let us stress here that due to variation of the metric along quasicylindrical ends not only the Laplace-Beltrami operator but also the Neumann boundary condition should be changed in PMLs. This new effect leads to significant difficulties in analysis of the PML method.

Relying on ideas of the Aguilar-Balslev-Combes-Simon theory of resonances [4, 16] we establish a limiting absorption principle. As is typically the case, scattered solutions satisfying the limiting absorption principle locally coincide with solutions to the problem with infinite PMLs. The latter solutions are of some exponential

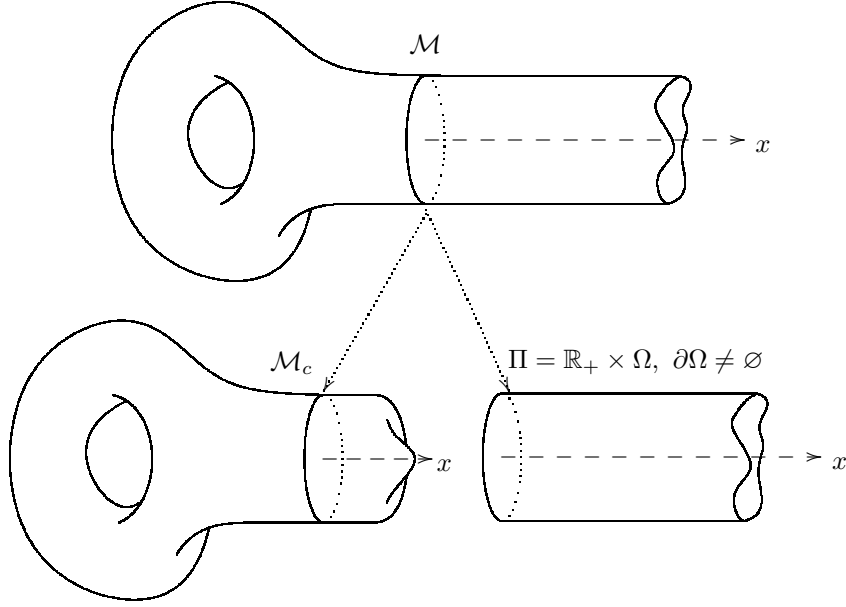


FIGURE 1. Representation $\mathcal{M} = \mathcal{M}_c \cup \Pi$.

decay at infinity. These results are mainly based on localization of the essential spectrum of non-selfadjoint operators corresponding to the problem with infinite PMLs. Thanks to the exponential decay of solutions in PMLs we can establish stability and exponential convergence of the PML method by using compound expansions. This is a further development of our scheme for analysis of stability and exponential convergence of the PML method [8, 7], see also [9]. The added difficulties are due to the new effect mentioned above. To overcome these difficulties we develop our approach to construction of PMLs and use non-homogeneous boundary value problems in localization of the essential spectrum and in compound expansions.

This paper is organized as follows. In Section 2 we introduce manifolds with axial analytic quasicylindrical ends and consider an illustrative example. Section 3 is devoted to construction of infinite PMLs. In Section 4 we localize the essential spectrum of the operator modeling infinite PMLs conjugated with exponent. In Section 5 we establish a limiting absorption principle and show that outgoing and incoming solutions are of some exponential decay in PMLs. Finally, in Section 6 we study the problem with finite PMLs and prove stability and exponential convergence of the PML method.

2. MANIFOLDS WITH AXIAL ANALYTIC QUASICYLINDRICAL ENDS

Let Ω be a compact (not necessarily simply connected) n -dimensional manifold with smooth boundary $\partial\Omega$. Denote by Π the semi-cylinder $\mathbb{R}_+ \times \Omega$, where \mathbb{R}_+ is the positive semi-axis, and \times stands for the Cartesian product. Consider an oriented connected $n+1$ -dimensional manifold \mathcal{M} representable in the form $\mathcal{M} = \mathcal{M}_c \cup \Pi$, where \mathcal{M}_c is a compact manifold with smooth boundary, see Fig. 1. We also assume that the boundary $\partial\mathcal{M}$ of \mathcal{M} is smooth.

Let \mathbf{g} be a Riemannian metric on \mathcal{M} . We identify the cotangent bundle $T^*\Pi$ with the tensor product $T^*\mathbb{R}_+ \otimes T^*\Omega$ via the natural isomorphism induced by the product structure on Π . Then

$$(1) \quad \mathbf{g}|_{\Pi} = \mathbf{g}_0 dx \otimes dx + 2\mathbf{g}_1 \otimes dx + \mathbf{g}_2, \quad \mathbf{g}_k(x) \in C^\infty T^*\Omega^{\otimes k}, \quad x \in \mathbb{R}_+.$$

Denote by $\mathbb{C}T^*\Omega^{\otimes k}$ the tensor power of the complexified cotangent bundle $\mathbb{C}T^*\Omega$ with fibers $\mathbb{C}T_y^*\Omega = T_y^*\Omega \otimes \mathbb{C}$. In what follows C^m stands for sections of complexified bundles. We equip the space $C^1 T^*\Omega^{\otimes k}$ with the norm

$$(2) \quad \|\cdot\|_\epsilon = \max_{y \in \Omega} (|\cdot|_\epsilon(y) + |\nabla \cdot|_\epsilon(y)),$$

where ϵ is a Riemannian metric on Ω , $|\cdot|_\epsilon(y)$ is the norm in $\mathbb{C}T_y^*\Omega^{\otimes k}$, and $\nabla : C^1T^*\Omega^{\otimes k} \rightarrow C^0T^*\Omega^{\otimes k+1}$ is the Levi-Civita connection on (Ω, ϵ) . The norms induced by different metrics ϵ are equivalent.

Definition 1. We say that $(\mathcal{M}, \mathbf{g})$ is a manifold with axial analytic quasicylindrical end $(\Pi, \mathbf{g} \upharpoonright_\Pi)$, if the following conditions hold:

- i. The coefficients $\mathbb{R}_+ \ni x \mapsto \mathbf{g}_k(x) \in C^\infty T^*\Omega^{\otimes k}$ in (1) extend by analyticity from the semi-axis \mathbb{R}_+ to the sector $\mathbb{S}_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha < \pi/4\}$.
- ii. The values $\|\mathbf{g}_0(z) - 1\|_\epsilon$, $\|\mathbf{g}_1(z)\|_\epsilon$, and $\|\mathbf{g}_2(z) - \mathbf{h}\|_\epsilon$ converge to zero uniformly in $z \in \mathbb{S}_\alpha$ as $|z| \rightarrow \infty$, where \mathbf{h} is a metric on Ω .

Long-range geometric perturbations of tubular waveguides (i.e. of manifolds $(\mathcal{M}, \mathbf{g})$ with $\mathbf{g} \upharpoonright_\Pi = dx \otimes dx + \mathbf{h}$) are included into consideration as there are no assumptions on the rate of convergence in condition ii of Definition 1. Let us give some illustrative examples of manifolds with axial analytic quasicylindrical ends.

Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary (Ω is a line segment if $n = 1$). By (x, y) and (s, t) we denote the Cartesian coordinates in \mathbb{R}^{n+1} , where $x, s \in \mathbb{R}$ and $y, t \in \mathbb{R}^n$. Consider a closed domain \mathcal{M} with smooth boundary such that $\{(x, y) \in \mathcal{M} : x \leq 0\}$ is a bounded subset of the half-space $\{(x, y) \in \mathbb{R}^{n+1} : x > -2\}$ and $\{(x, y) \in \mathcal{M} : x > 0\}$ is the semi-cylinder Π . Let

$$\mathcal{G} = \{(s, t) \in \mathbb{R}^{n+1} : (s, t) = \phi(x, y), (x, y) \in \mathcal{M}\}$$

be the image of \mathcal{M} under a diffeomorphism ϕ satisfying the following conditions:

- i. The function $x \mapsto \phi(x, \cdot) \in C^\infty(\Omega)$ has an analytic continuation from \mathbb{R}_+ to the sector \mathbb{S}_α ;
- ii. The element in the row ℓ and column m of the matrix $(\phi'(z, \cdot))^t \phi'(z, \cdot)$, where $(\phi')^t$ is the transpose of the Jacobian ϕ' , uniformly tends to the Kronecker delta $\delta_{\ell m}$ in the space $C^1(\Omega)$ as $|z|$ tends to infinity, $z \in \mathbb{S}_\alpha$.

For instance, we can take

$$\phi(x, y) = (x, (x+3)^\beta a + (1 + (x+3)^\gamma)y), \quad a \in \mathbb{R}^n, \beta < 1, \gamma < 0.$$

Then the boundary $\partial\mathcal{G}$ approaches at infinity the bent semi-cylinder $\{(s, t) : s \in \mathbb{R}_+, t - (s+3)^\beta a \in \Omega\}$. If we take

$$\phi(x, y) = \left(\int_0^x (1 + 1/\log(\tilde{x} + 4)) d\tilde{x}, (1 + 1/\log(x+5))y \right),$$

then $\partial\mathcal{G}$ slowly approaches at infinity the semi-cylinder $\mathbb{R}_+ \times \partial\Omega$, cf. Fig. 2. Time-harmonic acoustic

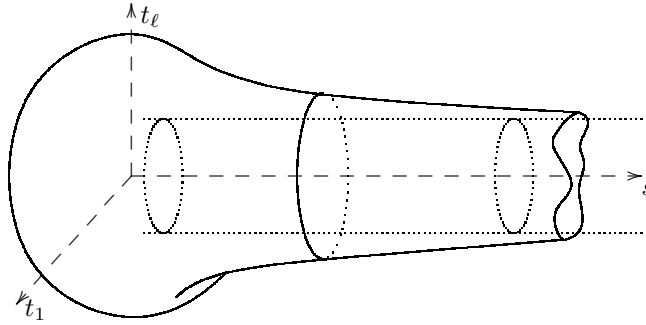


FIGURE 2. Domain $\mathcal{G} \subset \mathbb{R}^{n+1}$ with quasicylindrical end.

scattering at angular frequency ω in the domain \mathcal{G} filled by an inhomogeneous medium with anisotropic density tensor ρ and bulk modulus K is modeled by the Neumann Laplacian on $(\mathcal{G}, K\rho)$ with spectral parameter $\mu = \omega^2$, where $K\rho$ is a metric on \mathcal{G} , e.g., [3]. Under certain assumptions on regularity of ρ and K the pullback \mathbf{g} of $K\rho$ by the diffeomorphism ϕ satisfies the conditions of Definition 1. These assumptions are apriori met for homogeneous isotropic media, i.e. the pullback \mathbf{g} of the Euclidean metric $K\rho$ by the diffeomorphism ϕ satisfies the conditions of Definition 1. To study the Neumann Laplacian on $(\mathcal{G}, K\rho)$ is the same as to study the Neumann Laplacian on $(\mathcal{M}, \mathbf{g})$. For our aims it is natural and convenient to work on manifolds with axial analytic quasicylindrical ends.

3. CONSTRUCTION OF INFINITE PMLs

Our approach here is very close in spirit to [14]. Let $\mathbf{s}_r(x) = \mathbf{s}(x - r)$, where $r > 0$ is sufficiently large and $\mathbf{s} \in C^\infty(\mathbb{R})$ possesses the properties:

$$(3) \quad \begin{aligned} &\mathbf{s}(x) = 0 \text{ for all } x \leq 1, \\ &0 \leq \mathbf{s}'(x) \leq 1 \text{ for all } x \in \mathbb{R}, \text{ and } \mathbf{s}'(x) = 1 \text{ for large } x > 0; \end{aligned}$$

here $\mathbf{s}' = \partial \mathbf{s} / \partial x$. Let $T^* \mathbb{S}_\alpha$ be the holomorphic cotangent bundle $\{(z, c dz) : z \in \mathbb{S}_\alpha, c \in \mathbb{C}\}$, where $dz = d\Re z + i d\Im z$. Consider the tensor field

$$(4) \quad \mathbf{g}_0 dz \otimes dz + 2\mathbf{g}_1 \otimes dz + \mathbf{g}_2 \in C^\infty(T^* \mathbb{S}_\alpha \otimes T^* \Omega)^{\otimes 2}$$

with analytic coefficients $\mathbb{S}_\alpha \ni z \mapsto \mathbf{g}_k(z) \in C^\infty T^* \Omega^{\otimes k}$, cf. Definition 1. For all values of λ in the disk

$$(5) \quad \mathcal{O}_\alpha = \{\lambda \in \mathbb{C} : |\lambda| < \sin \alpha < 1/\sqrt{2}\}$$

the function \mathbf{s}_r defines the embedding

$$(6) \quad T^* \mathbb{R}_+ \ni (x, a dx) \mapsto (x + \lambda \mathbf{s}_r(x), a(1 + \lambda \mathbf{s}'_r(x))^{-1} dz) \in T^* \mathbb{S}_\alpha,$$

where $|1 + \lambda \mathbf{s}'_r(x)| > 1 - 1/\sqrt{2}$. This embedding together with (4) induces the tensor field

$$(7) \quad \mathbf{g}_\lambda \upharpoonright_\Pi = \mathbf{g}_{0,\lambda}^r dx \otimes dx + 2\mathbf{g}_{1,\lambda}^r \otimes dx + \mathbf{g}_{2,\lambda}^r \in C^\infty T^* \Pi^{\otimes 2},$$

where $\mathbf{g}_{k,\lambda}^r(x) \in C^\infty T^* \Omega^{\otimes k}$ are smooth in $x \in \mathbb{R}_+$ and analytic in $\lambda \in \mathcal{O}_\alpha$ coefficients given by

$$\mathbf{g}_{k,\lambda}^r(x) = (1 + \lambda \mathbf{s}'_r(x))^{2-k} \mathbf{g}_k(x + \lambda \mathbf{s}_r(x)).$$

Note that $\mathbf{g}_\lambda \upharpoonright_\Pi$ with $\lambda \in \mathcal{O}_\alpha \cap \mathbb{R}$ is the pullback of the metric $\mathbf{g} \upharpoonright_\Pi$ by the diffeomorphism $\varkappa_\lambda(x, y) = (x + \lambda \mathbf{s}_r(x), y)$ scaling the semi-cylinder Π along its axis \mathbb{R}_+ . Since \mathbf{s}_r is supported on (r, ∞) , the equality $\mathbf{g}_\lambda \upharpoonright_{(0,r) \times \Omega} = \mathbf{g} \upharpoonright_{(0,r) \times \Omega}$ holds for all $\lambda \in \mathcal{O}_\alpha$. We extend $\mathbf{g}_\lambda \upharpoonright_\Pi$ to \mathcal{M} by setting $\mathbf{g}_\lambda \upharpoonright_{\mathcal{M} \setminus \Pi} = \mathbf{g} \upharpoonright_{\mathcal{M} \setminus \Pi}$. As a result we obtain the analytic function

$$\mathcal{O}_\alpha \ni \lambda \mapsto \mathbf{g}_\lambda \in C^\infty T^* \mathcal{M}^{\otimes 2}.$$

Clearly, $\mathbf{g}_0 = \mathbf{g}$ and (7) with $\lambda = 0$ is the same as (1). By analyticity in λ we conclude that \mathbf{g}_λ is a symmetric tensor field. The Schwarz reflection principle gives $\overline{\mathbf{g}_\lambda} = \mathbf{g}_{\bar{\lambda}}$. It must be stressed that \mathbf{g}_λ with $\lambda \neq 0$ depends on r , however we do not indicate this for brevity of notations. As r is sufficiently large, \mathbf{g}_λ is non-degenerate; i.e. for $p \in \mathcal{M}$ and any nonzero $a \in \mathbb{C} T_p \mathcal{M}$ there exists $b \in \mathbb{C} T_p \mathcal{M}$ such that $\mathbf{g}_\lambda^p[a, b] \neq 0$, where $\mathbf{g}_\lambda^p[\cdot, \cdot]$ is the sesquilinear form naturally defined by \mathbf{g}_λ . Indeed, on $\mathcal{M} \setminus (r, \infty) \times \Omega$ the tensor field \mathbf{g}_λ is non-degenerate because it coincides there with the metric \mathbf{g} , while on $(r, \infty) \times \Omega$ it is only little different from the non-degenerate tensor field $(1 + \lambda \mathbf{s}'_r)^2 dx \otimes dx + \mathbf{h}$, see (7) and condition ii of Definition 1. In particular, $\mathbf{g}_\lambda \upharpoonright_\Pi$ coincides with $(1 + \lambda \mathbf{s}'_r)^2 dx \otimes dx + \mathbf{h}$ in the case of manifold with cylindrical end, i.e. in the case $\mathbf{g} \upharpoonright_\Pi = dx \otimes dx + \mathbf{h}$.

It is well known that a metric induces the musical isomorphism between tangent and cotangent bundles. Similarly, \mathbf{g}_λ induces the fiber isomorphism

$$\mathbb{C} T_p^* \mathcal{M} \ni \xi \mapsto {}^\lambda \sharp \xi \in \mathbb{C} T_p \mathcal{M},$$

where ${}^\lambda \sharp \xi$ is a unique vector satisfying the equality $\xi \bar{a} = \mathbf{g}_\lambda^p[{}^\lambda \sharp \xi, a]$ for all $a \in \mathbb{C} T_p \mathcal{M}$. We extend $\mathbf{g}_\lambda^p[\cdot, \cdot]$ to the pairs $(\xi, \eta) \in \mathbb{C} T_p^* \mathcal{M} \times \mathbb{C} T_p^* \mathcal{M}$ by setting

$$\mathbf{g}_\lambda^p[\xi, \eta] = \mathbf{g}_\lambda^p[{}^\lambda \sharp \xi, {}^{\bar{\lambda}} \sharp \eta], \quad \lambda \in \mathcal{O}_\alpha.$$

If λ is real, then $\mathbf{g}_\lambda^p[\cdot, \cdot]$ is the positive Hermitian form induced by the metric \mathbf{g}_λ . The corresponding volume form dvol_λ extends by analyticity to all $\lambda \in \mathcal{O}_\alpha$. Introduce the deformed global inner product

$$(\xi, \omega)_\lambda = \int_{\mathcal{M}} \mathbf{g}_\lambda[\xi, \omega] \text{dvol}_\lambda, \quad \lambda \in \mathcal{O}_\alpha, \quad \xi, \omega \in C_c^\infty T^* \mathcal{M}^{\otimes k}, \quad k = 0, 1.$$

Let us stress that for non-real $\lambda \in \mathcal{O}_\alpha$ the form dvol_λ is complex-valued and the deformed inner product $(\xi, \omega)_\lambda = \overline{(\omega, \xi)_\lambda}$ is not Hermitian. Let us consider the sesquilinear quadratic form

$$\mathbf{q}_\lambda[u, u] = (du, d({}^\lambda \sharp u))_\lambda, \quad u \in C_c^\infty(\mathcal{M}), \quad \lambda \in \mathcal{O}_\alpha,$$

where $d : C_c^\infty(\mathcal{M}) \rightarrow C_c^\infty T^* \mathcal{M}$ is the exterior derivative and $\varrho_\lambda \in C^\infty(\mathcal{M})$ is such that $\varrho_\lambda \text{dvol}_\lambda = \text{dvol}_0$.

Lemma 1. *Introduce the space $L^2(\mathcal{M})$ and the Sobolev space $H^1(\mathcal{M})$ as completion of the set $C_c^\infty(\mathcal{M})$ with respect to the norms*

$$\|u\| = \sqrt{(u, u)_0}, \quad \|u\|_{H^1(\mathcal{M})} = \sqrt{(du, du)_0 + (u, u)_0}$$

respectively. Then the family $\mathcal{O}_\alpha \ni \lambda \mapsto \mathbf{q}_\lambda$ of unbounded quadratic forms in $L^2(\mathcal{M})$ with domain $H^1(\mathcal{M})$ is analytic in the sense of Kato [10, 16]; i.e. \mathbf{q}_λ is a closed densely defined sectorial form, and the function $\mathcal{O}_\alpha \ni \lambda \mapsto \mathbf{q}_\lambda[u, u]$ is analytic for any $u \in H^1(\mathcal{M})$. Moreover, the sector of \mathbf{q}_λ is independent of $\lambda \in \mathcal{O}_\alpha$.

The family $\mathcal{O}_\alpha \ni \lambda \mapsto \mathbf{q}_\lambda$ uniquely determines an analytic family $\mathcal{O}_\alpha \ni \lambda \mapsto {}^\lambda\Delta$ of unbounded m-sectorial operators in $L^2(\mathcal{M})$ [10, 16]. (Here and elsewhere m-sectorial means that the numerical range $\{(Au, u) : u \in \mathcal{D}(A)\}$ and the spectrum $\sigma(A)$ of a closed unbounded operator A with the domain $\mathcal{D}(A)$ are both in some sector $\{\mu \in \mathbb{C} : |\arg(\mu + c)| \leq \vartheta < \pi/2, c > 0\}$.) We have

$$(8) \quad ({}^\lambda\Delta u, v)_0 = \mathbf{q}_\lambda[u, v], \quad u \in \mathcal{D}({}^\lambda\Delta), v \in H^1(\mathcal{M}).$$

In particular, $({}^0\Delta u, v)_0 = (du, dv)_0$ and ${}^0\Delta$ is the selfadjoint Neumann Laplacian. We consider ${}^\lambda\Delta$ with non-real $\lambda \in \mathcal{O}_\alpha$ as the operator modeling an infinite PML on $(r, \infty) \times \Omega$. We will show that this PML is an artificial nonreflective strongly absorbing layer for the outgoing (resp. incoming) solutions if $\Im \lambda > 0$ (resp. $\Im \lambda < 0$). Note that for $u \in \mathcal{D}({}^0\Delta)$ supported outside $(r, \infty) \times \Omega$ we have $u \in \mathcal{D}({}^\lambda\Delta)$ and ${}^0\Delta u \equiv {}^\lambda\Delta u$.

Proof. In a finite covering of Ω by coordinate neighborhoods $\{\mathcal{U}_j\}$ take a neighborhood \mathcal{U}_j and coordinates $y \in \mathbb{R}^n$ in \mathcal{U}_j . Then $\{\mathbb{R}_+ \times \mathcal{U}_j\}$ is a covering of Π and (x, y) are coordinates in $\mathbb{R}_+ \times \mathcal{U}_j$. Denote $\partial_0 = \partial/\partial x$, $\partial_\ell = \partial/\partial y_\ell$, $d_0 = dx$, and $d_\ell = dy_\ell$. As \mathbf{g}_λ is non-degenerate, the matrix $\mathbf{g}_{\lambda, \ell m}$ in the coordinate representation $\mathbf{g}_\lambda = \mathbf{g}_{\lambda, \ell m} d_\ell \otimes d_m$ has the inverse $\mathbf{g}_\lambda^{\ell m}$ and for $\xi = \xi_\ell d_\ell$ we have ${}^\lambda\sharp \xi = \mathbf{g}_\lambda^{\ell m} \xi_\ell \partial_m$. Hence $\mathbf{g}_\lambda[\xi, \xi] = \mathbf{g}_\lambda^{\ell m} \xi_\ell \bar{\xi}_m$.

On Ω all metrics are equivalent. In particular we can take $\epsilon = \delta_{\ell m} d_\ell \otimes d_m$, where $\delta_{\ell m}$ is the Kronecker delta. Then Definition 1 together with (7) immediately implies that

$$(9) \quad |\partial_k^j (\mathbf{g}_\lambda^{\ell m} - \text{diag}\{(1 + \lambda)^{-2}, \mathfrak{h}^{-1}\}_{\ell m})(x, y)| \leq C(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \quad j = 0, 1.$$

Moreover,

$$(10) \quad |\partial_k^j (\mathbf{g}_\lambda^{\ell m} - \text{diag}\{(1 + \lambda s'_r)^{-2}, \mathfrak{h}^{-1}\}_{\ell m})(x, y)| \leq \epsilon_r, \quad x \geq r, \quad \lambda \in \mathcal{O}_\alpha, \quad j = 0, 1,$$

with sufficiently small ϵ_r as r is sufficiently large. Since $\mathfrak{h}^{-1}(y)$ is a symmetric positive definite matrix, we get

$$\begin{aligned} |\arg((1 + \lambda s'_r(x))^{-2} |\xi_0|^2 + \mathfrak{h}^{\ell m}(y) \xi_\ell \bar{\xi}_m)| &< 2\alpha < \pi/2, \\ c|\xi|^2 &\leq |(1 + \lambda s'_r(x))^{-2} |\xi_0|^2 + \mathfrak{h}^{\ell m}(y) \xi_\ell \bar{\xi}_m| \leq |\xi|^2/c, \end{aligned}$$

where $|\xi|^2 = \xi_\ell \bar{\xi}_\ell$ and $c > 0$. This together with (10) gives

$$(11) \quad \delta|\xi|^2 \leq \Re(\mathbf{g}_\lambda^{\ell m}(x, y) \xi_\ell \bar{\xi}_m) \leq \delta|\xi|^2, \quad |\arg(\mathbf{g}_\lambda^{\ell m}(x, y) \xi_\ell \bar{\xi}_m)| \leq \vartheta,$$

where $x \geq r$, $\delta = \min\{c - \epsilon_r, (1/c + \epsilon_r)^{-1}\}$, and $\vartheta = 2\alpha + 2 \arcsin(\epsilon_r/2c) < \pi/2$.

From (11) and $\mathbf{g}_\lambda^p = \mathbf{g}^p$, $p \in \mathcal{M} \setminus (r, \infty) \times \Omega$, we conclude that the first term in the representation

$$(12) \quad \mathbf{q}_\lambda[u, u] = \int_{\mathcal{M}} \mathbf{g}_\lambda[du, du] \, \text{dvol}_0 + \int_{\mathcal{M}} \frac{\mathbf{g}_\lambda[du, u d\varrho_\lambda]}{\varrho_\lambda} \, \text{dvol}_0, \quad u \in C_c^\infty(\mathcal{M}),$$

meets the estimates

$$(13) \quad -\theta \leq \arg \int_{\mathcal{M}} \mathbf{g}_\lambda[du, du] \, \text{dvol}_0 \leq \theta, \quad c(du, du)_0 \leq \Re \int_{\mathcal{M}} \mathbf{g}_\lambda[du, du] \, \text{dvol}_0 \leq (du, du)_0/c$$

with some $\theta < \pi/2$ and $c > 0$. The second term has an arbitrary small relative bound with respect to the first term. Indeed,

$$(14) \quad \begin{aligned} \left| \int_{\mathcal{M}} \frac{\mathbf{g}_\lambda[du, u d\varrho_\lambda]}{\varrho_\lambda} \, \text{dvol}_0 \right| &\leq \frac{(1 + \tan \theta)}{\inf_{p \in \mathcal{M}} \varrho_\lambda(p)} \left(\Re \int_{\mathcal{M}} \mathbf{g}_\lambda[du, du] \, \text{dvol}_0 \right)^{1/2} \\ &\times \left(\Re \int_{\mathcal{M}} |u|^2 \mathbf{g}_\lambda[d\varrho_\lambda, d\varrho_\lambda] \, \text{dvol}_0 \right)^{1/2} \leq \epsilon \left| \int_{\mathcal{M}} \mathbf{g}_\lambda[du, du] \, \text{dvol}_0 \right| + \epsilon^{-1} C \|u\|^2 \end{aligned}$$

for any $\epsilon > 0$ and an independent of ϵ , λ , and u constant C . Here we used the uniform in λ and p estimates $|\varrho_\lambda(p)| \geq c > 0$ and $\mathbf{g}_\lambda^p[d\varrho_\lambda, d\varrho_\lambda] \leq C < \infty$, which are valid because $\varrho_\lambda(p) = 1$ for $p \notin (r, \infty) \times \Omega$ and $\varrho_\lambda = \sqrt{|\mathbf{g}_0|/|\mathbf{g}_\lambda|}$ in $\mathbb{R}_+ \times \mathcal{U}_j$, where $\partial_k^j |\mathbf{g}_\lambda|(x, y) \rightarrow \partial_k^j ((1 + \lambda \mathbf{s}'_r)^2 |\mathbf{h}|)(x, y)$ as $x \geq r \rightarrow \infty$, cf. (10) (as usual, $|\mathbf{g}| := \det(\mathbf{g}_{\ell m})$).

As a consequence of (12), (13), and (14) for some $\vartheta < \pi/2$ and some positive δ and γ we obtain

$$(15) \quad |\arg(\mathbf{q}_\lambda[u, u] + \gamma \|u\|^2)| \leq \vartheta, \quad \delta(du, du)_0 - \gamma \|u\|^2 \leq \Re \mathbf{q}_\lambda[u, u] \leq ((du, du)_0 + \|u\|^2)/\delta$$

uniformly in $\lambda \in \mathcal{O}_\alpha$. Therefore \mathbf{q}_λ in $L^2(\mathcal{M})$ with domain $H^1(\mathcal{M})$ is a closed densely defined sectorial form and its sector $\{\mu \in \mathbb{C} : |\arg(\mu + \gamma)| \leq \vartheta\}$ is independent of λ . By construction the function $\mathcal{O}_\alpha \ni \lambda \mapsto \mathbf{q}_\lambda[u, u]$ is analytic for any $u \in H^1(\mathcal{M})$. \square

4. CONJUGATED OPERATOR AND ITS ESSENTIAL SPECTRUM

In order to show exponential decay of outgoing (resp. incoming) solutions in infinite PMLs we study the operator ${}^\lambda \Delta$, $\Im \lambda > 0$ (resp. $\Im \lambda < 0$), conjugated with an exponent.

Let \mathbf{s} be a smooth function on the semi-cylinder Π , which depends only on the axial variable $x \in \mathbb{R}_+$ and possesses the properties (3). We extend \mathbf{s} to a smooth function on \mathcal{M} by setting $\mathbf{s}|_{\mathcal{M} \setminus \Pi} \equiv 0$. Consider the conjugated operator ${}^\lambda \Delta_\beta = e^{-\beta \mathbf{s}} {}^\lambda \Delta e^{\beta \mathbf{s}}$ with parameter $\beta \in \mathbb{C}$ on functions $\{u \in C_c^\infty(\mathcal{M}) : e^{\beta \mathbf{s}} u \in \mathcal{D}({}^\lambda \Delta)\}$, where $e^{\beta \mathbf{s}}$ is the operator of multiplication by the exponent. With ${}^\lambda \Delta_\beta$ we associate the quadratic form

$$\mathbf{q}_\lambda^\beta[u, u] = (d(e^{\beta \mathbf{s}} u), d(\overline{e^{-\beta \mathbf{s}} \varrho_\lambda u}))_\lambda, \quad u \in C_c^\infty(\mathcal{M}).$$

Lemma 2. *For any $\lambda \in \mathcal{O}_\alpha$ the form \mathbf{q}_λ^β in $L^2(\mathcal{M})$ is a closed sectorial form with the domain $H^1(\mathcal{M})$. Moreover, its sector is independent of λ .*

Proof. We have

$$\mathbf{q}_\lambda^\beta[u, u] - \mathbf{q}_\lambda[u, u] = \beta(u \, d\mathbf{s}, d(\overline{\varrho_\lambda u}))_\lambda - \beta(\varrho_\lambda \, du, u \, d\mathbf{s})_\lambda - \beta^2(\varrho_\lambda u \, d\mathbf{s}, u \, d\mathbf{s})_\lambda,$$

where the right hand side depends linearly on du . Similarly to (14) we conclude that the difference $\mathbf{q}_\lambda^\beta - \mathbf{q}_\lambda$ has an arbitrarily small relative bound with respect to \mathbf{q}_λ . More precisely,

$$|\mathbf{q}_\lambda^\beta[u, u] - \mathbf{q}_\lambda[u, u]| \leq \varepsilon |\mathbf{q}_\lambda[u, u]| + C(|\beta|, \varepsilon) \|u\|^2,$$

where $\varepsilon > 0$ is arbitrarily small and $C(|\beta|, \varepsilon)$ is independent of $\lambda \in \mathcal{O}_\alpha$ and $u \in C_c^\infty(\mathcal{M})$. This together with (15) completes the proof. \square

The Friedrichs extension of ${}^\lambda \Delta_\beta$ is an m-sectorial operator [10, 16]. Consider its domain $\mathcal{D}({}^\lambda \Delta_\beta)$ as a Hilbert space with the norm $\sqrt{\|\cdot\|^2 + \|{}^\lambda \Delta_\beta \cdot\|^2}$. We say that μ is a point of the essential spectrum $\sigma_{ess}({}^\lambda \Delta_\beta)$ if the bounded operator

$$(16) \quad {}^\lambda \Delta_\beta - \mu : \mathcal{D}({}^\lambda \Delta_\beta) \rightarrow L^2(\mathcal{M})$$

is not Fredholm. (Recall that a bounded linear operator is said to be Fredholm, if its kernel and cokernel are finite-dimensional, and the range is closed.)

Proposition 1. *Let $\lambda \in \mathcal{O}_\alpha$ and $\beta \in \mathbb{C}$. Then $\mu \in \sigma_{ess}({}^\lambda \Delta_\beta)$ if and only if*

$$(17) \quad \mu - (1 + \lambda)^{-2} (\xi + i\beta)^2 \in \sigma(\Delta_\Omega^N)$$

for some $\xi \in \mathbb{R}$, where $\sigma(\Delta_\Omega^N)$ is the spectrum of the selfadjoint Neumann Laplacian Δ_Ω^N on (Ω, \mathbf{h}) .

The spectrum $\sigma_{ess}({}^\lambda \Delta_\beta)$ is depicted on Fig. 3. In the case $\beta = 0$ the parabolas collapse to the dashed rays and we obtain the essential spectrum of ${}^\lambda \Delta \equiv {}^\lambda \Delta_0$. The proof of Proposition 1 is preceded by

Lemma 3. *The operator ${}^\lambda \Delta$, $\lambda \in \mathcal{O}_\alpha$, corresponds to a regular elliptic boundary value problem on \mathcal{M} .*

Proof. On $\mathcal{M} \setminus (r, \infty) \times \Omega$ the operator ${}^\lambda \Delta$ coincides with the Neumann Laplacian, which corresponds to a regular elliptic problem [15, 13].

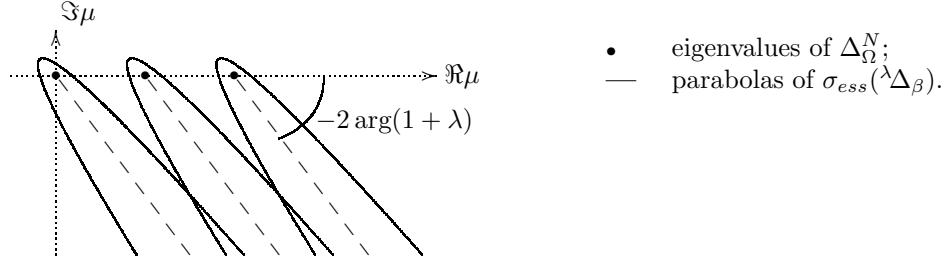


FIGURE 3. Essential spectrum of the operator ${}^\lambda\Delta_\beta$ for $\Im\lambda > 0$ and $\beta \geq 0$, the dashed rays correspond to $\beta = 0$.

Let $y \in \mathbb{R}^n$ be a system of coordinates in a neighborhood \mathcal{U}_j on Ω . In the case $\partial\Omega \cap \mathcal{U}_j \neq \emptyset$ we pick boundary normal coordinates $y = (y', y_n)$ such that ∂_{y_n} coincides with the unit inward normal derivative given by the metric \mathfrak{h} . From (8) it follows that in the coordinates (x, y) we have

$$(18) \quad {}^\lambda\Delta = -|\mathfrak{g}_\lambda|^{-1/2} \partial_\ell |\mathfrak{g}_\lambda|^{1/2} \mathfrak{g}_\lambda^{\ell m} \partial_m$$

and the functions $u \in \mathcal{D}({}^\lambda\Delta)$ satisfy some boundary condition ${}^\lambda\mathcal{N}u = 0$ on $\partial\mathcal{M}$ such that

$$(19) \quad {}^\lambda\mathcal{N}u = \mathfrak{g}_\lambda^{nm} \partial_m u \upharpoonright_{y_n=0} = 0$$

if $\partial\Omega \cap \mathcal{U}_j \neq \emptyset$; here notations are the same as in the proof of Lemma 1. The operator (18) is strongly elliptic for $x \geq r$ due to (11). It remains to show that the Shapiro-Lopatinskiĭ condition is met on $(r, \infty) \times (\partial\Omega \cap \mathcal{U}_j)$. Thanks to (10) the principal part of (18) for $y_n = 0$ (resp. the principal part of the operator of boundary conditions in (19)) is little different from $-(1 + \lambda s'_r(x))^{-2} \partial_0^2 - Q(y', \partial_{y'}) - \partial_n^2$ (resp. from ∂_n) uniformly in $x \geq r$, $\lambda \in \mathcal{D}_\alpha$, and y' as r is large. Here $-Q(y', \partial_{y'}) - \partial_n^2$ is the principal part of Δ_Ω . For $\xi = (\xi_0, \xi')$ on the unit sphere S^n and every bounded solution $u \neq 0$ of

$$((1 + \lambda s'_r(x))^{-2} \xi_0^2 + Q(y', \xi') - \partial_{y_n}^2) u(y_n) = 0 \text{ for } y_n \in \mathbb{R}_+$$

we have

$$\partial_{y_n} u(0) = C \partial_{y_n} e^{-y_n \sqrt{(1 + \lambda s'_r(x))^{-2} \xi_0^2 + Q(y', \xi')}} \upharpoonright_{y_n=0} = -C \sqrt{(1 + \lambda s'_r(x))^{-2} \xi_0^2 + Q(y', \xi')} \neq 0$$

because $\Re(1 + \lambda s'_r(x))^{-2} > 0$ and $Q(y', \xi')$ is a positive definite quadratic form in ξ' . In other words, the pair $(-(1 + \lambda s'_r(x))^{-2} \partial_0^2 + Q(y', \partial_{y'}) - \partial_n^2, \partial_n)$ satisfies the Shapiro-Lopatinskiĭ condition or, equivalently, the estimate

$$(20) \quad \|u\|_{H^2(\mathbb{R}_+)} \leq C (\|((1 + \lambda s'_r(x))^{-2} \xi_0^2 + Q(y', \xi') - \partial_{y_n}^2) u\|_{L^2(\mathbb{R}_+)} + |\partial_{y_n} u(0)|)$$

holds, e.g. [13, 15]. The constant C in (20) is independent of $r > 0$, $\xi \in S^n$, $\lambda \in \overline{\mathcal{O}_\alpha}$, $x \in \mathbb{R}_+$, and $y' \in \mathcal{U}_j \cap \partial\Omega$. (Indeed, it suffices to note that (20) with r replaced by r' can be obtained by the change of variables $x \mapsto x + r - r'$, the function s'_r varies over a compact subset of \mathbb{R}_+ only, and $(1 + \lambda s'_r(x))^{-2} \xi_0^2 + Q(y', \xi')$ is smooth in λ , ξ , x , and y' .) This together with (10) implies that the estimate similar to (20) is valid for the principal parts of (18) and (19) provided $x \geq r$, where r is sufficiently large. This completes the proof. \square

Now we are in position to prove Proposition 1.

Proof. We will rely on the following lemma due to Peetre, see e.g. [15, Lemma 5.1] or [13, Lemma 3.4.1]:

Let \mathcal{X}, \mathcal{Y} and \mathcal{Z} be Banach spaces, where \mathcal{X} is compactly embedded into \mathcal{Z} . Furthermore, let \mathcal{L} be a linear continuous operator from \mathcal{X} to \mathcal{Y} . Then the next two assertions are equivalent: (i) the range of \mathcal{L} is closed in \mathcal{Y} and $\dim \ker \mathcal{L} < \infty$, (ii) there exists a constant C such that

$$(21) \quad \|u\|_{\mathcal{X}} \leq C (\|\mathcal{L}u\|_{\mathcal{Y}} + \|u\|_{\mathcal{Z}}) \quad \forall u \in \mathcal{X}.$$

Sufficiency. Here we assume that μ does not satisfy (17) and establish an estimate of type (21) for the operator (16). Consider the operator $({}^\lambda\Delta, {}^\lambda\mathcal{N}) : C_c^\infty(\mathcal{M}) \rightarrow C_c^\infty(\mathcal{M}) \times C_c^\infty(\partial\mathcal{M})$ satisfying

$$({}^\lambda\Delta u, v)_0 + \langle {}^\lambda\mathcal{N}u, v \rangle = \mathfrak{q}_\lambda[u, v], \quad u \in C_c^\infty(\mathcal{M}), v \in H^1(\mathcal{M}),$$

where ${}^\lambda\mathcal{N}$ is the operator of boundary conditions on $\partial\mathcal{M}$ and $\langle \cdot, \cdot \rangle$ is the inner product on $(\partial\mathcal{M}, \mathbf{g}|_{\partial\mathcal{M}})$. Clearly, ${}^\lambda\mathcal{N}u = 0$ for $u \in \mathcal{D}({}^\lambda\Delta)$. Besides, in the local coordinates (x, y) on $\mathbb{R}_+ \times \Omega$ we have (18) and (19). In particular, ${}^0\Delta$ is the Laplace-Beltrami operator on $(\mathcal{M}, \mathbf{g})$ and ${}^0\mathcal{N}$ is the corresponding operator of the Neumann boundary conditions. Consider also the conjugated operator $({}^\lambda\Delta_\beta, {}^\lambda\mathcal{N}_\beta) = e^{-s\beta}({}^\lambda\Delta, {}^\lambda\mathcal{N})e^{s\beta}$.

Introduce the Sobolev space $H^\ell(\mathbb{R} \times \Omega)$ as the completion of the set $C_c^\infty(\mathbb{R} \times \Omega)$ with respect to the norm

$$\|u\|_{H^\ell(\mathbb{R} \times \Omega)} = \left(\int_{\mathbb{R}} \sum_{k \leq \ell} \|\partial_x^k u(x)\|_{H^{\ell-k}(\Omega)}^2 dx \right)^{1/2},$$

where $H^\ell(\Omega)$ is the Sobolev space on Ω . Let $H^{1/2}(\mathbb{R} \times \partial\Omega)$ be the space of traces $u|_{\mathbb{R} \times \partial\Omega}$ of the functions $u \in H^1(\mathbb{R} \times \Omega)$. Denote by ∂_η the operator of Neumann boundary conditions on $\mathbb{R} \times \partial\Omega$ taken with respect to the metric $dx \otimes dx + \mathbf{h}$. Applying the Fourier transform $\mathcal{F}_{x \rightarrow \xi}$ we pass from the continuous operator

$$(22) \quad (\Delta_\Omega - (1 + \lambda)^{-2}(\partial_x + \beta)^2 - \mu, \partial_\eta) : H^2(\mathbb{R} \times \Omega) \rightarrow H^0(\mathbb{R} \times \Omega) \times H^{1/2}(\mathbb{R} \times \partial\Omega)$$

to the operator $(\Delta_\Omega + (1 + \lambda)^{-2}(\xi + i\beta)^2 - \mu, \partial_\eta)$ of the Neumann problem on (Ω, \mathbf{h}) . As μ does not meet (17), the latter operator is invertible for all ξ and the inverse of (22) is given by

$$\mathcal{F}_{\xi \rightarrow x}^{-1} (\Delta_\Omega + (1 + \lambda)^{-2}(\xi + i\beta)^2 - \mu, \partial_\eta)^{-1} \mathcal{F}_{x \rightarrow \xi};$$

see e.g. [13, Theorem 5.2.2] or [12, Theorem 2.4.1] for details. As a consequence we have

$$(23) \quad \|u\|_{H^2(\mathbb{R} \times \Omega)} \leq C \|(\Delta_\Omega - (1 + \lambda)^{-2}(\partial_x + \beta)^2 - \mu, \partial_\eta)u\|_{H^0(\mathbb{R} \times \Omega) \times H^{1/2}(\mathbb{R} \times \partial\Omega)}.$$

Let $\chi_T(x) = \chi(x - T)$, where $\chi \in C^\infty(\mathbb{R})$ is a cutoff function such that $\chi(x) = 1$ for $x \geq 1$ and $\chi(x) = 0$ for $x \leq 0$. Then due to (9), (18), and (19) the constant $c(T)$ in the estimate

$$\|({}^\lambda\Delta_\beta - \Delta_\Omega + (1 + \lambda)^{-2}(\partial_x + \beta)^2, {}^\lambda\mathcal{N}_\beta - \partial_\eta)\chi_T u\|_{H^0(\mathbb{R} \times \Omega) \times H^{1/2}(\mathbb{R} \times \partial\Omega)} \leq c(T) \|\chi_T u\|_{H^2(\mathbb{R} \times \Omega)}$$

tends to zero as $T \rightarrow +\infty$. This together with (23) implies that for all sufficiently large T the estimate

$$(24) \quad \|\chi_T u\|_{H^2(\mathbb{R} \times \Omega)} \leq \mathbf{C} \|({}^\lambda\Delta_\beta - \mu, {}^\lambda\mathcal{N}_\beta)\chi_T u\|_{H^0(\mathbb{R} \times \Omega) \times H^{1/2}(\mathbb{R} \times \partial\Omega)}$$

holds, where $\mathbf{C} = (1/C - c(T))^{-1} > 0$.

Let $H^\ell(\mathcal{M})$ be the Sobolev space introduced as the completion of the set $C_c^\infty(\mathcal{M})$ in the norm

$$\|u\|_{H^\ell(\mathcal{M})} = \left(\|u\|_{H^\ell(\mathcal{M}_c)}^2 + \int_{\mathbb{R}_+} \sum_{k \leq \ell} \|\partial_x^k u(x)\|_{H^{\ell-k}(\Omega)}^2 dx \right)^{1/2},$$

where $\partial_x = \partial/\partial x$ and $H^\ell(\mathcal{M}_c)$ is the Sobolev space on the compact manifold \mathcal{M}_c . By $H^{1/2}(\partial\mathcal{M})$ we denote the space of traces on $\partial\mathcal{M}$ of functions in $H^1(\mathcal{M})$. We extend χ_T from its support in Π to \mathcal{M} by zero and rewrite (24) in the form

$$\|\chi_T u\|_{H^2(\mathcal{M})} \leq \mathbf{C} (\|\chi_T({}^\lambda\Delta_\beta - \mu, {}^\lambda\mathcal{N}_\beta)u\|_{L^2(\mathcal{M}) \times H^{1/2}(\mathcal{M})} + \|[({}^\lambda\Delta_\beta, {}^\lambda\mathcal{N}_\beta), \chi_T]u\|_{L^2(\mathcal{M}) \times H^{1/2}(\mathcal{M})}).$$

For the commutator we have $[({}^\lambda\Delta_\beta, {}^\lambda\mathcal{N}_\beta), \chi_T]u = [({}^\lambda\Delta_\beta, {}^\lambda\mathcal{N}_\beta), \chi_T](1 - \chi_{2T})u$ and therefore

$$\|[({}^\lambda\Delta_\beta, {}^\lambda\mathcal{N}_\beta), \chi_T]u\|_{L^2(\mathcal{M}) \times H^{1/2}(\mathcal{M})} \leq C \|(1 - \chi_{2T})u\|_{H^2(\mathcal{M})}.$$

Moreover, as a consequence of Lemma 3 the local elliptic coercive estimate

$$\|(1 - \chi_{2T})u\|_{H^2(\mathcal{M})} \leq C (\|(1 - \chi_{3T})({}^\lambda\Delta_\beta - \mu, {}^\lambda\mathcal{N}_\beta)u\|_{L^2(\mathcal{M}) \times H^{1/2}(\partial\mathcal{M})} + \|(1 - \chi_{3T})u\|)$$

is valid [15]. From the last three estimates it follows that

$$(25) \quad \|u\|_{H^2(\mathcal{M})} \leq C (\|({}^\lambda\Delta_\beta - \mu, {}^\lambda\mathcal{N}_\beta)u\|_{L^2(\mathcal{M}) \times H^{1/2}(\partial\mathcal{M})} + \|(1 - \chi_{3T})u\|).$$

In particular, (25) implies that $\|\cdot\|_{H^2(\mathcal{M})}$ is an equivalent norm in $\mathcal{D}({}^\lambda\Delta_\beta) = \{u \in H^2(\mathcal{M}) : {}^\lambda\mathcal{N}_\beta u = 0\}$.

Let \mathbf{w} be a bounded rapidly decreasing at infinity positive function on \mathcal{M} such that the embedding of $H^2(\mathcal{M})$ into the weighted space $L^2(\mathcal{M}, \mathbf{w})$ with the norm $\|\mathbf{w} \cdot\|$ is compact. Then $\mathcal{D}({}^\lambda\Delta_\beta)$ is compactly embedded into $L^2(\mathcal{M}, \mathbf{w})$ and (25) is an estimate of type (21) for the operator (16). Thus the range of (16) is closed and the kernel is finite-dimensional. In order to see that the cokernel is finite-dimensional, one can apply the same argument and obtain an estimate of type (21) for the adjoint operator.

Necessity. Now we assume that μ meets (17) for some j and show that the operator (16) is not Fredholm. It suffices to find a sequence $\{v_\ell\}_{\ell=1}^\infty$ in $\mathcal{D}({}^\lambda\Delta)$ violating the estimate (25).

We first show that for a regular point μ_0 of the m-sectorial operator ${}^\lambda\Delta_\beta$ the continuous operator

$$(26) \quad ({}^\lambda\Delta_\beta - \mu_0, {}^\lambda\mathcal{N}_\beta) : H^2(\mathcal{M}) \rightarrow L^2(\mathcal{M}) \times H^{1/2}(\partial\mathcal{M})$$

realizes an isomorphism. With this aim in mind we replace $(1 - \chi_{3T})u$ by wu in (25). Then by the Peetre lemma the range of (26) is closed. It is easy to see that the elements in the cokernel of the operator (26) are of the form $(v, v \upharpoonright_{\partial\mathcal{M}})$ with $v \in \ker({}^\lambda\Delta_\beta^* - \mu_0)$, where ${}^\lambda\Delta_\beta^*$ is adjoint to the m-sectorial operator ${}^\lambda\Delta_\beta$ in $L^2(\mathcal{M})$ with domain $\mathcal{D}({}^\lambda\Delta_\beta)$. Indeed, let (v, \underline{v}) be in the kernel of the adjoint operator

$$({}^\lambda\Delta_\beta - \mu_0, {}^\lambda\mathcal{N}_\beta)^* : L^2(\mathcal{M}) \times (H^{1/2}(\partial\mathcal{M}))^* \rightarrow (H^2(\mathcal{M}))^*.$$

Then $\langle {}^\lambda\Delta_\beta u - \mu_0 u, v \rangle_0 + \langle {}^\lambda\mathcal{N}_\beta u, \underline{v} \rangle = 0$ for all $u \in H^2(\mathcal{M})$, where $\langle \cdot, \cdot \rangle$ is extended to $H^{1/2}(\partial\mathcal{M}) \times (H^{1/2}(\partial\mathcal{M}))^*$. Since $\mathcal{D}({}^\lambda\Delta_\beta) \subset H^2(\mathcal{M})$, we immediately see that $v \in \ker({}^\lambda\Delta_\beta^* - \mu_0)$. Then for $u \in H^2(\mathcal{M})$ the Green identity gives $\langle {}^\lambda\mathcal{N}_\beta u, \underline{v} - v \upharpoonright_{\partial\mathcal{M}} \rangle = 0$ and therefore $\underline{v} = v \upharpoonright_{\partial\mathcal{M}}$. As a consequence, (26) is an isomorphism for $\mu_0 \notin \sigma({}^\lambda\Delta_\beta)$.

Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(x) = 1$ for $|x - 3| \leq 1$, and $\chi(x) = 0$ for $|x - 3| \geq 2$. Consider the functions

$$(27) \quad u_\ell(x, y) = \chi(x/\ell) \exp(i(1 + \lambda)\sqrt{\mu - \nu_j}x) \Phi(y), \quad (x, y) \in \mathbb{R} \times \Omega,$$

where Φ is an eigenfunction of the Neumann Laplacian Δ_Ω^N corresponding to the eigenvalue ν_j . It is clear that u_ℓ satisfies the Neumann boundary condition $\partial_\eta u_\ell = 0$ on $\mathbb{R} \times \partial\Omega$. As μ meets the condition (17), the exponent in (27) is an oscillating function of $x \in \mathbb{R}$. Straightforward calculation shows that

$$(28) \quad \|(\Delta_\Omega - (1 + \lambda)^{-2}(\partial_x + \beta)^2 - \mu)u_\ell\|_{H^0(\mathbb{R} \times \Omega)} \leq \text{const}, \quad \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)} \rightarrow \infty$$

as $\ell \rightarrow +\infty$. We extend the functions u_ℓ from Π to \mathcal{M} by zero and set

$$v_\ell = u_\ell - ({}^\lambda\Delta_\beta - \mu_0, {}^\lambda\mathcal{N}_\beta)^{-1}(0, {}^\lambda\mathcal{N}_\beta u_\ell),$$

where μ_0 is a regular point of the m-sectorial operator ${}^\lambda\Delta_\beta$. Clearly, $v_\ell \in \mathcal{D}({}^\lambda\Delta_\beta)$. We also have

$$(29) \quad \|({}^\lambda\Delta_\beta - \mu_0, {}^\lambda\mathcal{N}_\beta)^{-1}(0, {}^\lambda\mathcal{N}_\beta u_\ell)\|_{H^2(\mathcal{M})} \leq C \|({}^\lambda\mathcal{N}_\beta - \partial_\eta)u_\ell\|_{H^{1/2}(\partial\mathcal{M})} \leq \mathcal{C}_\ell \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)},$$

where $\mathcal{C}_\ell \rightarrow 0$ as $\ell \rightarrow +\infty$. Hence

$$(30) \quad \|v_\ell\|_{H^2(\mathcal{M})} \geq \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)} - \mathcal{C}_\ell \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)}, \quad \|v_\ell - u_\ell\|_{H^2(\mathcal{M})} \leq \mathcal{C}_\ell \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)}.$$

Assume that the estimate (25) is valid. Without loss of generality we can take a rapidly decreasing weight w such that $\|wu_\ell\| \leq \text{Const}$ uniformly in ℓ and the embedding $H^2(\mathcal{M}) \hookrightarrow L^2(\mathcal{M}; w)$ is compact. It is clear that $\|wu\| \leq c\|u\|_{H^2(\mathcal{M})}$ with some independent of $u \in H^2(\mathcal{M})$ constant c . Therefore

$$(31) \quad \|wv_\ell\| \leq \text{Const} + c\mathcal{C}_\ell \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)},$$

cf. (30). The estimate

$$\|({}^\lambda\Delta_\beta - \Delta_\Omega + (1 + \lambda)^{-2}(\partial_x + \beta)^2)u_\ell\| \leq \mathcal{C}_\ell \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)},$$

where $\mathcal{C}_\ell \rightarrow 0$ as $\ell \rightarrow +\infty$, together with (28) and (30) gives

$$(32) \quad \begin{aligned} \|({}^\lambda\Delta_\beta - \mu)v_\ell\| &\leq C \|(-(1 + \lambda)^{-2}(\partial_x + \beta)^2 + \Delta_\Omega - \mu)u_\ell\|_{H^0(\mathbb{R} \times \Omega)} \\ &+ \|({}^\lambda\Delta_\beta - \Delta_\Omega + (1 + \lambda)^{-2}(\partial_x + \beta)^2)u_\ell\| + \|({}^\lambda\Delta_\beta - \mu)(v_\ell - u_\ell)\| \\ &\leq C \cdot \text{const} + (\mathcal{C}_\ell + C\mathcal{C}_\ell) \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)}. \end{aligned}$$

Finally, as a consequence of (30), (25), (32) and (31), we get

$$(33) \quad \begin{aligned} \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)} - \mathcal{C}_\ell \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)} &\leq \|v_\ell\|_{H^2(\mathcal{M})} \leq C(\|({}^\lambda\Delta_\beta - \mu)v_\ell\| + \|wv_\ell\|) \\ &\leq C(C \cdot \text{const} + (C\mathcal{C}_\ell + \mathcal{C}_\ell) \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)} + \text{Const} + c\mathcal{C}_\ell \|u_\ell\|_{H^2(\mathbb{R} \times \Omega)}). \end{aligned}$$

Since $\mathcal{C}_\ell \rightarrow 0$ and $\mathcal{C}_\ell \rightarrow 0$, the inequalities (33) imply that the value $\|u_\ell\|_{H^2(\mathbb{R} \times \Omega)}$ remains bounded as $\ell \rightarrow +\infty$. This contradicts (28). Thus the sequence $\{v_\ell\}_{\ell=1}^\infty$ violates the estimate (25). \square

5. EXPONENTIAL DECAY OF SOLUTIONS IN INFINITE PMLs

Consider the algebra \mathcal{E} of all entire functions $\mathbb{C} \ni z \mapsto F(z, \cdot) \in C^\infty(\Omega)$ with the following property: in any sector $|\Im z| \leq (1-\epsilon)\Re z$ with $\epsilon > 0$ the value $\|F(z, \cdot)\|_{L^2(\Omega)}$ decays faster than any inverse power of $\Re z$ as $\Re z \rightarrow +\infty$. Examples of functions $F \in \mathcal{E}$ are $F(z, y) = e^{-\gamma z^2} P(z, y)$, where $\gamma > 0$ and $P(z, y)$ is an arbitrary polynomial in z with coefficients in $C^\infty(\Omega)$. We say that $f \in L^2(\mathcal{M})$ is an analytic vector, if $f(x, y) = F(x, y)$ for some $F \in \mathcal{E}$ and all $(x, y) \in \Pi$. For any f in the set \mathcal{A} of all analytic vectors we can define the analytic function $\mathcal{O}_\alpha \ni \lambda \mapsto f_\lambda \in L^2(\mathcal{M})$ by setting $f_\lambda = f$ on $\mathcal{M} \setminus (r, \infty) \times \Omega$ and $f_\lambda(x, y) = f(x + \lambda s_r(x), y)$ for $(x, y) \in (r, \infty) \times \Omega$. The set $\{f_\lambda : f \in \mathcal{A}\}$ is dense in $L^2(\mathcal{M})$ for any $\lambda \in \mathcal{D}_\alpha$, see, e.g., [6, Theorem 3].

Theorem 1. *Assume that $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega^N)$ is not an eigenvalue (i.e. not a trapped mode) of the self-adjoint Neumann Laplacian ${}^0\Delta$ in $L^2(\mathcal{M})$. Then the following assertions are valid.*

1. *For any $f \in \mathcal{A}$ there exist outgoing u_- and incoming u_+ solutions defined by the limiting absorption principle*

$$(34) \quad u_+ = \lim_{\epsilon \downarrow 0} ({}^0\Delta - \mu_0 - i\epsilon)^{-1} f, \quad u_- = \lim_{\epsilon \uparrow 0} ({}^0\Delta - \mu_0 - i\epsilon)^{-1} f,$$

where the limits are taken in the space $L_{loc}^2(\mathcal{M})$.

2. *For $\lambda \in \mathcal{O}_\alpha$ with $\Im \lambda < 0$ (resp. $\Im \lambda > 0$) the m -sectorial operator ${}^\lambda\Delta$ in $L^2(\mathcal{M})$ models PMLs on $(r, \infty) \times \Omega$ for incoming (resp. outgoing) solutions in the sense that $u_\lambda = ({}^\lambda\Delta - \mu_0)^{-1} f_\lambda$ coincides on $\mathcal{M} \setminus (r, \infty) \times \Omega$ with the incoming solution u_+ (resp. with the outgoing solution u_-) and*

$$(35) \quad \|e^{\beta s} u_\lambda\|_{H^2(\mathcal{M})} \leq C(\mu_0, \lambda) \|e^{\beta s} f_\lambda\|, \quad 0 \leq \beta < \min_{\nu \in \sigma(\Delta_\Omega^N)} |\Im\{(1+\lambda)\sqrt{\mu_0 - \nu}\}|.$$

The estimate (35) shows that u_λ decays exponentially in the PMLs.

Proof. 1. We need to prove that for any $\varrho \in C_c^\infty(\mathcal{M})$ and $f \in \mathcal{A}$ the function $\varrho({}^0\Delta - \mu_0 - i\epsilon)^{-1} f$ tends to some limits in $L^2(\mathcal{M})$ as $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$. Take a sufficiently large r such that $\text{supp } \varrho \cap (r, \infty) \times \Omega = \emptyset$. Let μ be outside the sector of m -sectorial operator ${}^\lambda\Delta$. Then for any real $\lambda \in \mathcal{O}_\alpha$ the change of variable $x \mapsto x + \lambda s_r(x)$ outside $\text{supp } \varrho$ implies

$$(36) \quad \varrho({}^0\Delta - \mu)^{-1} f = \varrho({}^\lambda\Delta - \mu)^{-1} f_\lambda.$$

From Lemma 1 and (8) it follows that the resolvent $({}^\lambda\Delta - \mu)^{-1}$ is an analytic function of $\lambda \in \mathcal{O}_\alpha$ with values in the space of bounded operators in $L^2(\mathcal{M})$ [16, Theorem XII.7]. Thus (36) extends by analyticity to all $\lambda \in \mathcal{O}_\alpha$. As μ is a regular point of ${}^\lambda\Delta$ in the simply connected set $\mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$ (see Proposition 1), the Fredholm analytic theory implies that the resolvent

$$\mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta) \ni \mu \mapsto ({}^\lambda\Delta - \mu)^{-1} : L^2(\mathcal{M}) \rightarrow \mathcal{D}({}^\lambda\Delta)$$

is a meromorphic operator function; see, e.g., [12, Appendix]. It remains to show that $\mu_0 \in \mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$ is not a pole. Then the right hand side of (36) with $\Im \lambda < 0$ (resp. $\Im \lambda > 0$) provides the left hand side with analytic continuation from $\mu = \mu_0 + i\epsilon$ to $\mu = \mu_0$ as $\epsilon \downarrow 0$ (resp. $\epsilon \uparrow 0$).

For μ outside of the sector of ${}^\lambda\Delta$ and real $\lambda \in \mathcal{O}_\alpha$ by the change of variable $x \mapsto x + \lambda s_r(x)$ we obtain

$$(37) \quad (({}^0\Delta - \mu)^{-1} f, g)_0 = (({}^\lambda\Delta - \mu)^{-1} f_\lambda, g_\lambda)_\lambda, \quad f, g \in \mathcal{A}.$$

This equality extends by analyticity to all $\lambda \in \mathcal{O}_\alpha$. Suppose, by contradiction, that μ_0 is a pole of $({}^\lambda\Delta - \mu)^{-1}$ with $\Im \lambda \leq 0$. As the sets $\{f_\lambda : f \in \mathcal{A}\}$ and $\{g_\lambda : g \in \mathcal{A}\}$ are dense in $L^2(\mathcal{M})$, the right hand side of (37) has a pole at μ_0 for some f and g . Then (37) implies that $(P_{\mu_0} f, g) \neq 0$, where P_{μ_0} is the projection onto the eigenspace of the selfadjoint operator ${}^0\Delta$, and thus $\ker(\Delta - \mu_0) \neq \{0\}$. This is a contradiction.

2. Since (36) is valid for any $\varrho \in C_c^\infty(\mathcal{M})$ such that $\text{supp } \varrho \cap (r, \infty) \times \Omega = \emptyset$, our construction in the proof of assertion 1 shows that u_λ coincides on $\mathcal{M} \setminus (r, \infty) \times \Omega$ with u_+ (resp. u_-) if $\Im \lambda < 0$ (resp. $\Im \lambda > 0$). The condition on β in (35) guarantees that μ_0 together with all points outside of the sector of m -sectorial operator ${}^\lambda\Delta_\beta$ is in the simply connected subset of $\mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta_\beta)$, see Proposition 1. Then the Fredholm analytic theory implies that μ_0 is either an eigenvalue of ${}^\lambda\Delta_\beta$ or $\mu_0 \notin \sigma({}^\lambda\Delta_\beta)$, e.g., [12, Appendix]. The inclusion $\Psi \in \ker({}^\lambda\Delta_\beta - \mu_0)$ gives $e^{-\beta s} \Psi \in \ker({}^\lambda\Delta - \mu_0) = \{0\}$ as μ_0 is not a pole of $({}^\lambda\Delta - \mu)^{-1}$. Hence $\mu_0 \notin \sigma({}^\lambda\Delta_\beta)$. This together with the equality $e^{\beta s} u_\lambda = ({}^\lambda\Delta_\beta - \mu_0)^{-1} e^{\beta s} f_\lambda$ justifies the estimate (35), cf. (25). \square

6. FINITE PMLs, STABILITY AND EXPONENTIAL CONVERGENCE OF THE PML METHOD

Consider the compact manifold $\mathcal{M}_R = \mathcal{M} \setminus (R, \infty) \times \Omega$. The boundary $\partial\mathcal{M}_R$ of \mathcal{M}_R is piecewise smooth. It has two conic points in the case of a 1-dimensional manifold Ω and the edge $\partial\Omega \times \{R\}$ otherwise. We denote $\partial\mathcal{M}_R \setminus (\{R\} \times \Omega) = \partial\mathcal{M}_R^-$. Introduce the Sobolev space $H^2(\mathcal{M}_R)$ as the completion of the set $C_c^\infty(\mathcal{M}_R)$ with respect to the norm

$$\|u\|_{H^2(\mathcal{M}_R)} = \left(\|u\|_{H^2(\mathcal{M}_c)}^2 + \sum_{\ell \leq 2} \int_0^R \|\partial_x^\ell u(x)\|_{H^{2-\ell}(\Omega)}^2 dx \right)^{1/2},$$

where $H^2(\mathcal{M}_c)$ and $H^{2-\ell}(\Omega)$ are the Sobolev spaces on the smooth compact manifolds \mathcal{M}_c and Ω . Consider the problem with finite PMLs: *given $F \in L^2(\mathcal{M}_R)$ find a solution $v \in H^2(\mathcal{M}_R)$ to the problem*

$$(38) \quad (\lambda\Delta - \mu_0)v = F \text{ on } \mathcal{M}_R; \quad \lambda\mathcal{N}v = 0 \text{ on } \partial\mathcal{M}_R^-; \quad v = 0 \text{ on } \{R\} \times \Omega.$$

In the next theorem we prove a stability result for this problem.

Theorem 2. *Let $\lambda \in \mathcal{O}_\alpha \setminus \mathbb{R}$. Assume that $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega)$ is not an eigenvalue of the selfadjoint Neumann Laplacian ${}^0\Delta$ in $L^2(\mathcal{M})$. Then there exists a large number $R_0 > 0$ such that for all $R > R_0$ and $F \in L^2(\mathcal{M}_R)$, (38) has a unique solution $v \in H^2(\mathcal{M}_R)$. Moreover, the estimate*

$$(39) \quad \|v\|_{H^2(\mathcal{M}_R)} \leq C\|F\|_{L^2(\mathcal{M}_R)}$$

holds with independent of $R > R_0$ and F constant C .

In the proof of Theorem 2 we rely on compound expansions. This requires construction of an approximate solution to (38) compounded of solutions to limit problems, e.g., [13]. Being substituted into (38) the approximate solution leaves a discrepancy, which tends to zero as R increases. In contrast to the case of the Dirichlet Laplacian [7], here the discrepancy left in the boundary conditions on $\partial\mathcal{M}_R^-$ cannot be made small for large R if we use homogeneous limit problems. As the first limit problem we take the problem with infinite PMLs and non-homogeneous boundary conditions on $\partial\mathcal{M}$. In the next lemma we study the second limit problem.

Introduce the weighted Sobolev space $H_\beta^k((-\infty, R) \times \Omega)$ as the completion of the set $C_c^\infty((-\infty, R) \times \Omega)$ in the norm

$$\|u\|_{H_\beta^k((-\infty, R) \times \Omega)} = \left(\sum_{\ell \leq k} \int_{-\infty}^R \|e^{-\beta x} \partial_x^\ell u(x)\|_{H^{k-\ell}(\Omega)}^2 dx \right)^{1/2}.$$

We also set $L_\beta^2((-\infty, R) \times \Omega) = H_\beta^0((-\infty, R) \times \Omega)$ and denote by $H_\beta^{1/2}((-\infty, R) \times \partial\Omega)$ the space of traces on $(-\infty, R) \times \partial\Omega$ of the functions in $H_\beta^1((-\infty, R) \times \Omega)$.

Lemma 4. *Assume that $\lambda \in \mathcal{O}_\alpha \setminus \mathbb{R}$, $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega^N)$, and β is in the interval (35). Then for any $f \in L_\beta^2((-\infty, R) \times \Omega)$ and $g \in H_\beta^{1/2}((-\infty, R) \times \partial\Omega)$ there exists a solution u to the second limit problem*

$$(40) \quad (\Delta_\Omega - (1 + \lambda)^{-2} \partial_x^2 - \mu_0)u = f \text{ on } (-\infty, R) \times \Omega, \quad \partial_\eta u = g \text{ on } (-\infty, R) \times \partial\Omega, \quad u = 0 \text{ on } \{R\} \times \Omega,$$

satisfying the estimate

$$(41) \quad \|u\|_{H_\beta^2((-\infty, R) \times \Omega)} \leq C(\|f\|_{L_\beta^2((-\infty, R) \times \Omega)} + \|g\|_{H_\beta^{1/2}((-\infty, R) \times \partial\Omega)}),$$

where the constant C is independent of f and g . Here ∂_η is the operator of the Neumann boundary conditions on $(-\infty, R) \times \partial\Omega$ induced by the product metric $dx \otimes dx + \mathfrak{h}$ on $(-\infty, R) \times \Omega$.

Proof. Without loss of generality we can assume that $R = 0$ (the general case can be obtained by the change of variables $x \mapsto x - R$). We give only a sketch of the proof as it is essentially based on a well known argument, details can be found e.g. in [13].

Assume that $(f, g) \in C_c^\infty(\mathbb{R}_- \times \Omega) \times C_c^\infty(\mathbb{R}_- \times \partial\Omega)$ and extend it to a function in $C_c^\infty(\mathbb{R} \times \Omega) \times C_c^\infty(\mathbb{R} \times \partial\Omega)$ by setting $(f, g)(-x) = -(f, g)(x)$ for $x < 0$. The Fourier transform $\mathcal{F}_{x \mapsto \xi}(f, g)$ is an entire function of ξ with values in $L^2(\Omega) \times H^{1/2}(\partial\Omega)$ decaying faster than $|\xi|^{-k}$ with any $k > 0$ as ξ tends at infinity in any strip $|\Im \xi| < C$. Since $\mu_0 - (1 + \lambda)^{-2} \xi^2$ with $0 \leq \Im \xi < \beta$ is not an eigenvalue of Δ_Ω^N and the strongly elliptic

operator $\Delta_\Omega - (1 + \lambda)^{-2} \partial_x^2$ on $\mathbb{R} \times \Omega$ with the operator ∂_η on $\mathbb{R} \times \partial\Omega$ set up a regular elliptic problem, the elliptic coercive estimate

$$(42) \quad \sum_{p=0}^2 |\xi|^{2p} \|\Psi\|_{H^{2-p}(\Omega)}^2 \leq C (\|((1 + \lambda)^{-2} \xi^2 + \Delta_\Omega - \mu) \Psi\|_{L^2(\Omega)}^2 + \|\xi\| \|\partial_\eta \Psi\|_{L^2(\partial\Omega)}^2 + \|\partial_\eta \Psi\|_{H^{1/2}(\partial\Omega)}^2)$$

with parameter $0 \leq \Im \xi < \beta$ is valid. Moreover, the operator

$$(\Delta_\Omega + (1 + \lambda)^{-2} \xi^2 - \mu_0, \partial_\eta) : H^2(\Omega) \rightarrow H^0(\Omega) \times H^{1/2}(\partial\Omega), \quad 0 \leq \Im \xi < \beta,$$

yields an isomorphism and

$$(43) \quad u(x) = \mathcal{F}_{\xi \mapsto x}^{-1} (\Delta_\Omega + (1 + \lambda)^{-2} \xi^2 - \mu_0, \partial_\eta)^{-1} \mathcal{F}_{x \mapsto \xi}(\mathbf{f}, \mathbf{g})$$

is a unique in $L^2(\mathbb{R} \times \Omega)$ solution to the Neumann problem

$$(\Delta_\Omega - (1 + \lambda)^{-2} \partial_x^2 - \mu_0)u = \mathbf{f} \text{ on } \mathbb{R} \times \Omega, \quad \partial_\eta u = \mathbf{g} \text{ on } \mathbb{R} \times \partial\Omega.$$

Usual argument on smoothness of solutions to elliptic problems gives $u \in C^\infty(\mathbb{R} \times \Omega)$. Since $x \mapsto (\mathbf{f}, \mathbf{g})(x)$ is odd it follows that $\mathcal{F}_{x \mapsto \xi}(\mathbf{f}, \mathbf{g})$ is an odd function of ξ . Therefore $x \mapsto u(x)$ is odd and $u(0) = 0$.

The Cauchy integral theorem and the estimate (42) allow to replace the contour of integration \mathbb{R} in (43) by $\{\xi \in \mathbb{C} : \Im \xi = \beta\}$. Then the Parseval equality implies that u is a solution to the problem (40) satisfying the estimate (41). By continuity our construction extends to all $\mathbf{f} \in L_\beta^2(\mathbb{R}_- \times \Omega)$ and $\mathbf{g} \in H_\beta^{1/2}(\mathbb{R}_- \times \partial\Omega)$. \square

Now we are in position to prove Theorem 2.

Proof. The proof is carried out using the compound expansion method. We say that $w \in H^2(\mathcal{M}_R)$ is an approximate solution to the non-homogeneous problem

$$(44) \quad (\lambda \Delta - \mu_0)v = F \text{ on } \mathcal{M}_R, \quad \lambda \mathcal{N}v = G \text{ on } \partial \mathcal{M}_R^-, \quad v = 0 \text{ on } \{R\} \times \Omega$$

if the following conditions are satisfied:

i. The estimate

$$\|w\|_{H^2(\mathcal{M}_R)} \leq c \|(F, G)\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial \mathcal{M}_R^-)}$$

holds with an independent of F , G , and R constant c ;

ii. The estimate

$$(45) \quad \|(\lambda \Delta - \mu_0, \lambda \mathcal{N})w - (F, G)\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial \mathcal{M}_R^-)} \leq C_R \|(F, G)\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial \mathcal{M}_R^-)}$$

is valid, where the constant C_R is independent of F and G and $C_R \rightarrow 0$ as $R \rightarrow +\infty$.

Due to Condition i w_R continuously depends on F and G . Condition ii implies that the discrepancy, left by w_R in the problem (44), tends to zero as $R \rightarrow +\infty$. Once an approximate solution is found, it is not hard to verify the assertion of the theorem.

Let $\rho \in C^\infty(\mathbb{R})$ be a cutoff function such that $\rho(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1/2$. We set $\rho_R = \rho(x - R)$ on Π and $\varrho_R = 1$ on $\mathcal{M} \setminus \Pi$. Let $(f, g) = \varrho_{R/2}(F, G)$ and $(\mathbf{f}, \mathbf{g}) = (1 - \varrho_R)(F, G)$. Extend (F, G) from $\mathcal{M}_R \times \partial \mathcal{M}_R^-$ to $\mathcal{M} \times \partial \mathcal{M}$ and (\mathbf{f}, \mathbf{g}) from $(0, R) \times \Omega \times (0, R) \times \partial\Omega$ to $(-\infty, R) \times \Omega \times (-\infty, R) \times \partial\Omega$ by zero. We already know that $\mu_0 \notin \sigma(\lambda \Delta_\beta)$ and hence (26) yields an isomorphism for β in the interval in (35) (see the proof of Theorem 1.2). We find an approximate solution w compounded of $u_\lambda = e^{-\beta s}(\lambda \Delta_\beta - \mu_0, \lambda \mathcal{N}_\beta)^{-1} e^{\beta s}(f, g)$ and a solution $u \in H_\beta^2((-\infty, R) \times \Omega)$ to the equation (40) in the form

$$w = \varrho_R u_\lambda + (1 - \varrho_{R/3})u;$$

here the second term in the right hand side is extended from $(-\infty, R) \times \Omega$ to \mathcal{M}_R by zero. On the support of (f, g) we have $e^{\beta s} \leq C e^{\beta R/2}$ and on the support of (\mathbf{f}, \mathbf{g}) we have $e^{-\beta x} \leq C e^{-\beta R/2}$ uniformly in R . Hence

$$(46) \quad \begin{aligned} & \|e^{\beta s}(f, g)\|_{L^2(\mathcal{M}) \times H^{1/2}(\partial \mathcal{M})} + e^{\beta R} \|(\mathbf{f}, \mathbf{g})\|_{H_\beta^0((-\infty, R) \times \Omega) \times H_\beta^{1/2}((-\infty, R) \times \partial\Omega)} \\ & \leq C e^{\beta R/2} \|(F, G)\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial \mathcal{M}_R^-)} \end{aligned}$$

with an independent of R and (F, G) constant C . Thanks to (41) and (46) we can conclude that

$$\|w\|_{H^2(\mathcal{M}_R)}^2 \leq \|\varrho_R u_\lambda\|_{H^2(\mathcal{M})}^2 + \|(1 - \varrho_{R/3})u\|_{H^2((-\infty, R) \times \Omega)}^2 \leq C \|(F, G)\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial \mathcal{M}_R^-)}^2$$

and Condition i is satisfied.

Let us verify Condition ii. We have

$$(47) \quad \begin{aligned} (\lambda\Delta - \mu_0, \lambda\mathcal{N})w - (F, G) &= [(\lambda\Delta, \lambda\mathcal{N}), \varrho_R]u_\lambda + ((1 + \lambda)^{-2}[\partial_x^2, \rho_{R/3}]u, 0) \\ &\quad + (\lambda\Delta - \Delta_\Omega + (1 + \lambda)^{-2}\partial_x^2, \lambda\mathcal{N} - \partial_\eta)(1 - \rho_{R/3})u. \end{aligned}$$

On the support of the commutator $[(\lambda\Delta, \lambda\mathcal{N}), \varrho_R]u_\lambda$ the weight $e^{\beta s}$ is bounded from below by $ce^{\beta R}$ uniformly in $R > 0$. As a consequence,

$$(48) \quad \|[(\lambda\Delta, \lambda\mathcal{N}), \varrho_R]u_\lambda\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial\mathcal{M}_R^-)} \leq C_1 e^{-\beta R} \|e^{\beta s} u_\lambda\|_{H^2(\mathcal{M})} \leq C_2 e^{-\beta R} \|e^{\beta s}(f, g)\|_{L^2(\mathcal{M}) \times H^{1/2}(\partial\mathcal{M})}.$$

Now we estimate the second term in the right hand side of (47). On the support of $[\partial_x^2, \rho_{R/3}]u$ we have $e^{-\beta x} \geq Ce^{-\beta R/3}$. Thanks to (41) we obtain

$$(49) \quad \begin{aligned} \|(1 + \lambda)^{-2}[\partial_x^2, \rho_{R/3}]u\|_{L^2(\mathcal{M}_R)} &\leq C_1 e^{\beta R/3} \|u; H_\beta^2((-\infty, R) \times \Omega)\| \\ &\leq C_2 e^{\beta R/3} \|(\mathbf{f}, \mathbf{g})\|_{H_\beta^0((-\infty, R) \times \Omega) \times H_\beta^{1/2}((-\infty, R) \times \Omega)}. \end{aligned}$$

Finally, consider the last term in the right hand side of (47). On the support of $(1 - \rho_{R/3})u$ the coefficients of the operator $(\lambda\Delta - \Delta_\Omega + (1 + \lambda)^{-2}\partial_x^2, \lambda\mathcal{N} - \partial_\eta)$ uniformly tend to zero as $R \rightarrow +\infty$; see (9) and (18), (19). This together with (41) gives

$$(50) \quad \begin{aligned} \|(\lambda\Delta - \Delta_\Omega + (1 + \lambda)^{-2}\partial_x^2, \lambda\mathcal{N} - \partial_\eta)(1 - \rho_{R/3})u\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial\mathcal{M}_R^-)} \\ \leq c_R \|(\mathbf{f}, \mathbf{g})\|_{H_0^0((-\infty, R) \times \Omega) \times H_0^{1/2}((-\infty, R) \times \Omega)}, \end{aligned}$$

where $c_R \rightarrow 0$ as $R \rightarrow +\infty$. From (46)–(50) it follows that w meets Condition ii. Thus w is an approximate solution to the problem (44).

Now we are in position to prove the assertion of the theorem. Observe that $(\lambda\Delta, \lambda\mathcal{N})w - (f, g) = \mathfrak{D}_R(f, g)$ with some operator \mathfrak{D}_R in $L^2(\mathcal{M}_R) \times H^{1/2}(\partial\mathcal{M}_R^-)$, whose norm $\|\mathfrak{D}_R\|$ tends to zero as $R \rightarrow +\infty$ because of Condition ii on w . For all $R > R_0$ we have $\|\mathfrak{D}_R\| \leq \epsilon < 1$. Hence there exists the inverse $(I + \mathfrak{D}_R)^{-1}$ and its norm is bounded by the constant $1/(1 - \epsilon)$ uniformly in $R > R_0$. We set $(\tilde{f}, \tilde{g}) = (I + \mathfrak{D}_R)^{-1}(f, g)$. In the same way as before we construct the approximate solution w for the problem (44), where (f, g) is replaced by (\tilde{f}, \tilde{g}) . Then for $v = w$ we have $(\lambda\Delta - \mu_0, \lambda\mathcal{N})v = (\tilde{f}, \tilde{g}) + \mathfrak{D}_R(\tilde{f}, \tilde{g}) = (f, g)$ and

$$\|v; H^2(\mathcal{M}_R)\| \leq c \|(\tilde{f}, \tilde{g})\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial\mathcal{M}_R^-)} \leq c/(1 - \epsilon) \|(f, g)\|_{L^2(\mathcal{M}_R) \times H^{1/2}(\partial\mathcal{M}_R^-)},$$

where C is independent of $R > R_0$. In particular for $R > R_0$ and $f \in L^2(\mathcal{M}_R)$ there exists a solution $v \in H^2(\mathcal{M}_R)$ to the problem (38) satisfying (39). This solution is unique as a similar argument shows that the adjoint problem is solvable in $H^2(\mathcal{M}_R)$ for any right hand side in $L^2(\mathcal{M}_R)$. \square

In contrast to infinite PMLs, finite PMLs are not non-reflective. Reflections produce a non-zero difference in \mathcal{M}_r between solutions u_\pm satisfying the limiting absorption principle (34) and solutions $v \in H^2(\mathcal{M}_R)$ to the problem (38) with $F = f_\lambda$ and $\Im\lambda \leq 0$. In the next theorem we prove that this difference (error) decays with an exponential rate as $R \rightarrow +\infty$. In other words, we prove exponential convergence of the PML method. The problem with finite PMLs can be solved numerically with the help of finite element solvers; discretization produces yet another error that we do not estimate here.

Theorem 3. *Let $\lambda \in \mathcal{O}_\alpha \setminus \mathbb{R}$ and let β be in the interval in (35). Assume that $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega^N)$ is not an eigenvalue of the selfadjoint Neumann Laplacian ${}^0\Delta$ in $L^2(\mathcal{M})$. Then there exists $R_0 > 0$ such that for any $f \in \mathcal{A}$ and for all $R > R_0$ a unique solution $v_R \in H_0^2(\mathcal{M}_R)$ of the problem (38) with $F = f_\lambda$ converges on \mathcal{M}_r to the outgoing solution u_- (resp. the incoming solution u_+) of the equation $({}^0\Delta - \mu_0)u = f$ in the case $\Im\lambda > 0$ (resp. $\Im\lambda < 0$) in the sense that as $R \rightarrow +\infty$ the estimate*

$$(51) \quad \|u_\pm - v_R\|_{H^2(\mathcal{M}_r)}^2 \leq Ce^{-2\beta R} \|e^{\beta s} f_\lambda\|^2$$

holds with a constant C independent of $R > R_0$ and f .

Let us recall that the set \mathcal{A} of analytic functions is dense in $L^2(\mathcal{M})$. In particular, our construction and Theorem 3 remain valid for any $f \in L^2(\mathcal{M})$ supported in \mathcal{M}_r .

Proof. By Theorem 1 it suffices to prove the estimate (51) with u_{\pm} replaced by u_{λ} . Let $\rho \in C^{\infty}(\mathbb{R})$ be a cutoff function such that $\rho(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1/2$. We set $\rho_R(x) = \rho(x - R)$ on Π and $\rho_R = 1$ on $\mathcal{M} \setminus \Pi$. Then $\rho_R u_{\lambda} - v_R = u_{\lambda} - v_R$ on \mathcal{M}_R and the difference $\rho_R u_{\lambda} - v_R \in H^2(\mathcal{M}_R)$ satisfies (38) with $F = (\rho_R - 1)f_{\lambda} + [\lambda \Delta, \rho_R]u_{\lambda}$. Observe that

$$\|(\rho_R - 1)f_{\lambda}\|_{L^2(\mathcal{M}_R)} \leq C e^{-\beta R} \|e^{\beta s} f_{\lambda}\|, \quad \|[\lambda \Delta, \rho_R]u_{\lambda}\|_{L^2(\mathcal{M}_R)} \leq C e^{-\beta R} \|e^{\beta s} u_{\lambda}\|_{H^2(\mathcal{M})}.$$

This together with (35) gives

$$(52) \quad \|F\|_{L^2(\mathcal{M}_R)} \leq c e^{-\beta R} \|e^{\beta s} f_{\lambda}\|.$$

By Theorem 2 we have

$$\|\rho_R u_{\lambda} - v_R\|_{H^2(\mathcal{M}_R)} \leq C \|F\|_{L^2(\mathcal{M}_R)}, \quad R > R_0.$$

This together with

$$\|u_{\pm} - v_R\|_{L^2(\mathcal{M}_r)} \leq \|\rho_R u_{\lambda} - v_R\|_{H^2(\mathcal{M}_R)}, \quad R > R_0 > r,$$

and (52) completes the proof of (51). \square

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REFERENCES

- [1] J. P. Bérenger, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comp. Phys., 114 (1994), pp. 185–200.
- [2] F. Collino, P. Monk, *The perfectly matched layer in curvilinear coordinates*, SIAM J. Sci. Comput., 19 (1998), pp. 2061–2090.
- [3] S. A. Cummer, M. Rahm, and D. Schurig *Material parameters and vector scaling in transformation acoustics*, New Journal of Physics 10 (2008) 115025, 12pp
- [4] H. L. CYCON, R. G. FROESE, W. KIRSCH, AND B. SIMON, *Schrödinger operators, with application to quantum mechanics and global geometry*, Springer-Verlag, New York, 1986.
- [5] T. Hohage, F. Schmidt, and L. Zschiedrich, *Solving time-harmonic scattering problems based on the pole condition II. Convergence of the PML method*, SIAM J. Math. Anal., 35 (2003), pp. 547–560.
- [6] W. Hunziker, *Distortion analyticity and molecular resonance curves*. Ann. Inst. H. Poincaré Phys. Theor. 45 (1986) 339–358.
- [7] V. Kalvin, *Limiting absorption principle and perfectly matched layer method for Dirichlet Laplacians in quasi-cylindrical domains*, SIAM J. Math. Anal. 44 (2012), 355–382.
- [8] V. Kalvin, *Perfectly matched layers for diffraction gratings in inhomogeneous media. Stability and error estimates*, SIAM J. Numer. Anal. 49 (2011), 309–330.
- [9] V. Kalvin, *Spectral deformations for quasicylindrical domains*, Commun. Contemp. Math. (2013) DOI: 10.1142/S0219199712500654
- [10] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin-Heidelberg-New York, 1966.
- [11] S. Kim, J.E. Pasciak, *Analysis of the spectrum of a Cartesian Perfectly Matched Layer (PML) approximation to acoustic scattering problem*, J. Math. Anal. Appl. 361 (2010), 420–430; *Analysis of a Cartesian PML approximation to acoustic scattering problems in \mathbb{R}^2* , J. Math. Anal. Appl. 370 (2010), 168–186.
- [12] V. A. Kozlov, V. G. Maz'ya, *Differential equations with operator coefficients (with applications to boundary value problems for partial differential equations)*, Berlin, Springer-Verlag, 1999.
- [13] V. A. Kozlov, V. G. Maz'ya, J. Rossmann, *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs, vol. 52, American Mathematical Society, 1997.
- [14] M. Lassas, J. Liukkonen, E. Somersalo, *Complex Riemannian metric and absorbing boundary conditions*, J. Math. Pures Appl., 80 (2001), 739–768.
- [15] J.-L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications I,II*, Springer-Verlag, New York-Heidelberg, 1972.
- [16] M. Reed, B. Simon, *Methods of modern mathematical physics I-IV*. Academic Press, New York, 1972.

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